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NPS55Mt75101

NAVAL POSTGRADUATE SCHOOL

Monterey, California

MANPOWER PLANNING MODELS - V

OPTIMIZATION MODELS

by

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and

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October 1975

Approved for public release; distribution unlimited.

Prepared for:

Navy Personnel R&D Center
San Diego, California

NAVAL POSTGRADUATE SCHOOL
Monterey, California

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This work was supported in part by the Navy Personnel R&D Center, San Diego, the Office of Naval Research and the Manpower Planning Division (MP120) of the Marine Corps.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

(14)

1. REPORT NUMBER: NPS55Mt75101
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER

(6)

4. TITLE (and Subtitle): Manpower Planning Models - V. Optimization Models.

(9)

5. ~~TYPE OF REPORT & PERIOD COVERED~~: Technical Report,
6. ~~PERFORMING ORG. REPORT NUMBER~~

(10)

7. AUTHOR(s): R. C. Grinold, K. T. Marshall
8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School, Monterey, California 93940
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: N6822176WR60008
11. CONTROLLING OFFICE NAME AND ADDRESS: Navy Personnel R&D Center, San Diego, California

(11)

12. ~~REPORT DATE~~: Octo[redacted] 1975

(12) 67p.

13. ~~NUMBER OF PAGES~~
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Manpower	Modelling
Planning	Optimization

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is the fifth in a series of reports on Manpower Planning Models. The emphasis in this report is on optimization in both short and long term planning. Models discussed in earlier reports form the basis of the optimization models. A detailed illustrative example is included to demonstrate how the models may be applied to policy determination for a university faculty.

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S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

251 450

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V. OPTIMIZATION MODELS

1. Introduction.

This chapter presents optimization models based on the manpower flow processes described in previous chapters. It is not possible or desirable to describe the entire range of optimization models. Each organization has its own singular features, objectives, and difficulties. There can be no all encompassing model that would address the problems of every organization. Therefore, this chapter will emphasize modeling techniques and the use of optimization models in fashioning manpower policy. We stress the role of aggregate planning and design problems. We avoid the operational problems of personnel management. For example, an aggregate university model might identify three types of students (graduates, upper division, and lower division) and three types of teachers (tenure, non-tenure, and teaching assistants). An operational planning model would have a finer breakdown of the student and teaching components, and would be concerned with detailed problems of matching the teaching resources to the demand for instruction.

The most important features of manpower models are the relatively long time horizons involved with manpower decisions and the uncertainty in future manpower requirements. This chapter addresses these problems directly, and presents some approximation techniques that can be used to explore the range of possible manpower policies.

The optimization problems described in this chapter are a part of the planning process. They are not intended to dominate that process. The models are intended as calculating devices to assist in the evaluation of policy. A decision maker inputs data and assumptions to the optimization model and obtains a unique, and in one sense optimal, specification of future system performance.

The data may consist of projected legacies, future requirements, budget conditions, costs, discount rates, utilization factors, and the coefficients governing the flow process. The data itself represents assumptions about the future and in many respects is a specification of future manpower policy. The decision maker feeds the data to the optimization problem and is presented with a description of future performance. The planner is then free to change the input data and explore a broad range of policy alternatives. If the planning process is viewed this way, then the optimization problems become the heart of a policy simulation or policy exploration model.

The material in this chapter is necessarily technical. Each section starts with a brief resume of its results. The non-technical reader should focus on these introductions. Section 3 is an example of the use of optimization procedures in designing an organization. The non-technical reader should have no difficulty with Section 3. In general, we retain the notation and conventions developed in earlier chapters. We do allow the index of the longitudinal matrices $P(u)$ to run through all non-negative integers to avoid unnecessarily complicated notation resulting from the previous maximum M. We exploit the probabilistic interpretation of the flow process whenever it eases the exposition. The more technical sections assume a familiarity with linear programming and duality theory.

Section 2 examines a long-run optimization model based on a longitudinal flow process. We are able to calculate approximately optimal operating policies for the control of this system through the indefinite future. An example is included in Section 3. The example uses the techniques developed in Section 2 to design a faculty promotion and hiring system. Section 4 indicates how the long-run model can be used in conjunction with short run planning constraints.

This type of analysis is particularly useful in cases of expansion or drastic institutional change. It is necessary to exercise a greater degree of control during the early transition phase than is desirable when the system settles into its assigned role.

In Section 5, we examine the special single class-single chain model and derive correspondingly sharper results. Section 6 described a procedure for treating problems with uncertain requirements and other uncertain aspects of the flow process. The technique allows one to calculate reasonable immediate decisions and to obtain an estimate of the long run impact of the uncertainty on system performance.

2. Optimal Long Run Operations.

This section examines the problem of determining optimal, or at least good, long range operating policies for manpower planning models. This is an important consideration for manpower planners since the effects of one period's planning decisions will be felt over a large number of future periods. The succeeding sections show how this long run approach can be combined with a detailed analysis of a short planning horizon. This combined approach allows us to focus on the important current planning decisions without sacrificing the long term view.

The linear programming model presented in this section uses the inflows to the manpower system as decision variables and minimization of discounted cost as the objective. The data which defines the flow process, costs, and constraints are presumed to be given. We are aware that many important policy variables must be selected in order to define this data. The data reflect a combination of policy decisions, behavioral traits, and economic parameters beyond the control of the system. Our optimization procedure should be viewed in this light. Given the data, and the policy implicit in the data, there is still a great deal of choice in the way the organization can be run.

To eliminate that choice, and therefore, to relate the policy directly to performance, we choose the least cost method of operation. The idea behind using such a model is to determine and compare the results of alternative policies. Fortunately, the calculating device of linear programming is helpful in this regard. We obtain a substantial amount of information about the effects of changes in input data on system performance, and, if the model is small enough, it is not too difficult to resolve the problem frequently using different sets of data. Therefore, we are not presenting the linear program as a "take it or leave it" policy maker. The linear program should be viewed as a tool for exploring the connection between policy alternatives and system performance.

This section is necessarily more technical than most parts of the book. Readers not trained in the techniques of linear programming will probably obtain a sharper understanding of the procedure by first reading the application in Section 3 and checking back to this section to see formal framework.

In our model we make several simplyfying assumptions. Most of these can be eased to some degree, but these extensions are left as problems for the interested reader. The assumptions are

(1)

(i) The manpower flow process can be described by a longitudinal model.

(ii) The variable costs of operating the system are proportional to stocks and flows.

(iii) The system is of constant size.

(iv) Future costs are discounted at rate $\alpha < 1$.

(v) The constraints on stocks and flows are homogeneous. That is, if g_1 and g_2 satisfy the flow constraints then $g_1 + g_2$ and λg_1 where $\lambda \geq 0$ is a scalar) will also satisfy the flow constraints.

The basic equation of longitudinal flow gives the stocks at time t,

(2) $$s(t) = \sum_{j=1}^{t} P(t-j)g(j) + \ell(t),$$

where $\ell(t)$ is the legacy of decisions prior to period 1. The size of the manpower system is $\sum_{i=1}^{N} s_i(t) = es(t)$. If we assume $es(0) = \rho$, then our constant size restriction is

(3) $$e \sum_{j=1}^{t} P(t-j)g(j) = \rho - e\ell(t), \qquad t \geqq 1.$$

The homogeneous constraints on manpower stocks can be written

(4) $$As(t) \geqq 0, \qquad t \geq 1.$$

With the aid of (2), this becomes

$$A \sum_{j=1}^{t} P(t-j)g(j) \geqq -A\ell(t), \qquad t \geqq 1. \tag{5}$$

The restriction to homogeneous constraints (4), is not as serious a limitation as it might at first appear.

Example 1. Suppose we would like a weighted sum of stocks in a period to be at least λ. Then if h_i is the weight applied to stock i, $hs(t) = \sum_{i=1}^{N} h_i s_i(t) \geqq \lambda$. This is equivalent to

$$\frac{hs(t)}{es(t)} \geq \frac{\lambda}{\rho}$$

or

$$(h - e\,\frac{\lambda}{\rho})s(t) \geqq 0.$$

Hence, the original constraint can be written in this homogeneous form.

Problem 1. Suppose $s(t)$ must satisfy a constraint of the form $\frac{hs(t)}{fs(t)} \leq \lambda$, where $es(t) = \rho > 0$, and $s(t) \geq 0$ insure that $fs(t) > 0$ Show this is equivalent to the constraint $(\lambda f-h)s(t) \geq 0$. □

The constraints on flows are

$$Bg(t) \geqq 0, \quad g(t) \geqq 0, \quad \text{for} \quad t \geqq 1. \tag{6}$$

Let the N-vector $a = (a_1,\ldots,a_N)$ and the K-vector $b = (b_1,b_2,\ldots,b_K)$ give the costs of supporting stock and flow respectively in a given period. The present (time zero) value of the cost incurred in period t (and realized at time t) is

$$\alpha^t[as(t)+bg(t)].$$

The total cost of operating the system in the indefinite future is

$$\sum_{t=1}^{\infty} \alpha^{t}[as(t)+bg(t)]. \tag{7}$$

We shall use (2) to eliminate $s(t)$ from this expression, but first we prove a basic result which is needed in this chapter, namely

$$\sum_{t=1}^{\infty} \alpha^{t}\Big[\sum_{j=1}^{t} P(t-j)g(j)\Big] = \Big(\sum_{u=0}^{\infty} \alpha^{u}P(u)\Big) \sum_{t=1}^{\infty} \alpha^{t}g(t). \tag{8}$$

We can depict the first sum in (8) in the triangular form

$$\begin{array}{lr}
\;\;\alpha\,(P(0)g(1) &) \\
+\,\alpha^{2}(P(1)g(1) + P(0)g(2) &) \\
+\,\alpha^{3}(P(2)g(1) + P(1)g(2) + P(0)g(3) &) \\
+\,\alpha^{4}(P(3)g(1) + P(2)g(2) + P(1)g(3) + P(0)g(4) &) \\
\;\;\vdots \quad \vdots & \\
\end{array}$$

Summing the rows of this array yields the total on the left of (8), while summing the columns yields the right hand side of (8).

Now let

$$\tilde{P}(\alpha) = \sum_{u=0}^{\infty} \alpha^{u}P(u).$$

Using these results we combine (2) and (6) to obtain

$$\sum_{t=1}^{\infty} a^{t}[as(t)+bg(t)] = c\sum_{t=1}^{\infty} \alpha^{t}g(t) + a\tilde{\ell}(\alpha) \tag{9}$$

where $c = a\tilde{P}(\alpha)+b$ and $\tilde{\ell}(\alpha) = \sum_{t=1}^{\infty} \alpha^{t}\ell(t)$.

Notice that the cost component $(a\tilde{\ell}(\alpha))$ can be considered as a <u>sunk</u> cost since it depends on legacies of past inputs. The next section, however, indicates how policies that change the legacy can be evaluated. In that case the component $a\tilde{\ell}(\alpha)$ is important.

Problem 2: Suppose the cost of supporting stock $s_i(t)$ depends upon the length of service distribution of that stock. Let $a_i(u)$ be the cost of support for an individual in class i with length of service equal to u. The present value of supporting future stocks is

$$v_i = \sum_{t=1}^{\infty} \alpha^t \sum_{u=0}^{t-1} a_i(u) s_i(t;u),$$

where $s_i(t;u)$ is the number of individuals in class i at time t with length of service equal to u.

From Chapter II, Section 2, Equation 1,

$$s_i(t;u) = \sum_{k=1}^{K} p_{ik}(u) q_k(t-u).$$

Show that v_i, above can be rewritten as

$$v_i = \sum_{t=1}^{\infty} \alpha^t \tilde{a}_i(\alpha) g(t),$$

where the k-th element of $\tilde{a}_i(\alpha)$ is

$$\tilde{a}_{ik}(\alpha) = \sum_{u=0}^{\infty} \alpha^u a_i(u) p_{ik}(u).$$

Therefore

$$\sum_{t=1}^{N} v_i = \sum_{t=0}^{\infty} \alpha^t \left(\sum_{i=1}^{N} \tilde{a}_i(\alpha) \right) g(t) = \tilde{a}(\alpha) \sum_{t=0}^{\infty} \alpha^t \; g(t).$$

The total cost is as before with $c = \tilde{a}(\alpha) + b$. □

The problem of selecting optimal inflow vectors $g(1), g(2), \ldots, g(t), \ldots$ is obtained from (3), (5), (6) and (9):

$$\text{Minimize} \quad c \sum_{t=1}^{\infty} \alpha^t g(t)$$

$$\text{subject to} \quad e \sum_{j=1}^{t} P(t-j) g(j) = \rho - e\ell(t),$$

$$(10) \qquad A \sum_{j=1}^{t} P(t-j)g(j) \geqq -A\ell(t),$$

$$Bg(t) \geqq 0,$$

$$g(t) \geqq 0,$$

$$\text{for} \quad t \geqq 1.$$

The linear program (10) has an infinite number of constraints and variables. In general, we cannot hope to obtain an exact solution of (10), but the techniques presented in this section indicate how an approximately optimal solution can be obtained by solving a linear program with a relatively small number of constraints and variables. We first derive a linear program (11) from (10) with the property that any feasible solution $\{g(t)\}$ of (10) determines a feasible solution $g = \sum_{t=1}^{\infty} \alpha^t g(t)$ of (11) with equal objective value. It follows that the optimal value of the solution of (11) is then a lower bound on the optimal value of (10). We next use the optimal solution of (11) to construct a policy (a sequence $\{g^*(t)\}$) that actually achieves this lower bound That solution is approximately optimal in the sense that it (on the average) satisfies the constraints of (10) and achieves the minimum cost lower bound.

We derive the finite linear program (11) from (10) by multiplying the t-th set of constraints by α^t and summing. Recall from equation (8), that

$$\sum_{t=1}^{\infty} \alpha^t \sum_{j=1}^{t} P(t-j)g(j) = \tilde{P}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t).$$

This result is the key to transforming (10). The linear program is

$$\text{Minimize} \quad cg$$

$$(11) \qquad \text{subject to} \quad A\tilde{P}(\alpha)g \geqq -A\tilde{\ell}(\alpha)$$

$$e\tilde{P}(\alpha)g = \frac{\alpha\rho}{1-\alpha} - e\tilde{\ell}(\alpha)$$

$$Bg \geqq 0, \qquad g \geqq 0.$$

This program has only K (the number of chains) variables, and is usually very easily solved.

For any sequence $\{g(t)\}$ that is feasible for (10) the variable $g = \sum_{t=1}^{\infty} \alpha^t g(t)$ is feasible for (11), and (11) and (10) have exactly the same objective value. The infinite sum will always converge due to the constraint $es(t) = \rho$, for all t. This observation shows that the optimal value of (11) is a lower bound on the optimal value of (10). It is possible that the optimal solution g^* of (11) is such that no solution of (10), $\{g(t)\}$, exists with $\sum_{t=1}^{\infty} \alpha^t g(t) = g^*$. In that case the lower bound is slack and the optimal value of (11) is strictly lower than that of (10). If there does exist a feasible solution $\{g(t)\}$ of (10) such that $\sum_{t=1}^{\infty} \alpha^t g(t) = g^*$, then the bound is exact and $\{g(t)\}$ is an optimal solution of (10).

Suppose we solve (11) and obtain an optimal solution g^*. We now indicate how to compute an "approximately" optimal solution of (10) from this optimal solution g^* of (11). The constructed solution, call it $\{g^*(t)\}$, is not guaranteed to be feasible for (10). It will satisfy the constraints $Bg(t) \geqq 0$, $g(t) \geqq 0$, and $es(t) = \rho$, and it will <u>attain</u> the lower bound on the optimal value of (10). However, we cannot guarantee that it will satisfy the constraints $As(t) \geqq 0$ for all t; it will satisfy these constraints in a weighted average sense, and one can easily check to see if the equilibrium solution implied by $\{g^*(t)\}$ satisfies the stock constraints $As(t) \geqq 0$ for large values of t.

The solution $\{g^*(t)\}$ will be of the form $\{\gamma(t)g\}$ where $\gamma(t)$ is a scalar calculated by solving the following lower triangular system of equations.

$$
\begin{aligned}
&p(0)\gamma(1) && = r(1) \\
&p(1)\gamma(1) + p(0)\gamma(2) && = r(2) \\
&\vdots && \\
&\sum_{j=1}^{t} p(t-j)\gamma(j) && = r(t).
\end{aligned} \tag{12}
$$

In (12) $p(u) = eP(u)g^*$ and $r(t) = \rho - e\ell(t)$. These equations are selected to assure $g^*(t)$ will satisfy the size constraints, since

$$e \sum_{j=1}^{t} P(t-j)\gamma(j)g^* = \sum_{j=1}^{t} p(t-j)\gamma(j) = r(t) = \rho - e\ell(t).$$

In section 5 we give general conditions on $p(u)$ and $r(t)$ that imply $\gamma(j) \geqq 0$ for all j. In our specific case we have the total legacy $e\ell(t)$ decreasing. Therefore, $r(t)$ is increasing. By assumption $eP(u) \geqq eP(u+1)$ since the vector $eP(u)$ gives the probability of staying in the system u or more periods for each chain. The solution g^* is nonnegative, so $p(u) = eP(u)g^* \geqq p(u+1) = eP(u+1)g^*$. By subtracting the t-th equation of (12) from the $t+1$-th we obtain

$$p(0)\gamma(t+1) = [r(t+1) - r(t)] + \sum_{j=1}^{t} [p(t-j) - p(t+1-j)]\gamma(j). \tag{13}$$

If $r(t)$ is increasing, $p(u)$ decreasing, $p(0) > 0$, and $\gamma(1),\ldots,\gamma(t) \geqq 0$, then (13) implies that $\gamma(t+1) \geqq 0$. It follows by induction that $\gamma(t) \geqq 0$ for <u>all</u> t. This assures $Bg^*(t) = \gamma(t)Bg^* \geqq 0$ and $g^*(t) = \gamma(t)g^* \geqq 0$.

Note that

$$\sum_{t=1}^{\infty} \alpha^t g^*(t) = \left[\sum_{t+1}^{\infty} \alpha^t \gamma(t)\right] g^* = \mu g^*,$$

where we have let $\sum_{t=1}^{\infty} \alpha^t \gamma(t) = \mu$. The cost of the program $\{g^*(t)\}$ is equal to $\mu c g^*$. We now show that $\mu = 1$, and therefore the program $\{g^*(t)\}$ <u>attains</u> the lower bound cg^* on the optimal value of (10). From the feasibility of g^* in (11) we have

$$e\tilde{P}(\alpha)g^* = \frac{\rho\alpha}{1-\alpha} - e\tilde{\ell}(\alpha). \tag{14}$$

If we multiply the t-th constraint of (12) by α^t and sum we obtain (recall (8))

$$\left(\sum_{j=0}^{\infty} \alpha^u p(u)\right)\left(\sum_{t=1}^{\infty} \alpha^t \gamma(t)\right) = \sum_{t=1}^{\infty} \alpha^t r(t) = \frac{\alpha\rho}{1-\alpha} - \sum_{t=1} \alpha^t e\ell(t).$$

However $\sum_{j=0}^{\infty} \alpha^u p(u) = e\tilde{P}(\alpha)g^*$, and $\sum_{t=1}^{\infty} \alpha^t e\ell(t) = e\tilde{\ell}(\alpha)$. Therefore,

$$e\tilde{P}(\alpha)g^*\left(\sum_{t=1}^{\infty} \alpha^t \gamma(t)\right) = \frac{\rho\alpha}{1-\alpha} - e\tilde{\ell}(\alpha). \tag{15}$$

A comparison of (14) and (15) forces us to conclude that $\mu = \sum_{t=1}^{\infty} \alpha^t \gamma(t) = 1$.

Finally, we turn our attention to the stock constraints. Let

$$s^*(t) = \ell(t) + \sum_{j=1}^{t} P(t-j)g^*(t).$$

There is no guarantee that $As^*(t) \geqq 0$ for all t. However, the constraints are satisfied in a weighted sense. The weight for the t-th constraint being $\frac{\alpha^{t-1}}{1-\alpha}$. The sum

$$\sum_{t=1}^{\infty} \frac{\alpha^{t-1}}{1-\alpha} As^*(t) = \frac{1}{\alpha(1-\alpha)} [A\tilde{P}(\alpha)g^* + A\tilde{\ell}(\alpha)] \geqq 0.$$

Notice that the weights place stronger emphasis on the earlier periods, since $\frac{\alpha^{t-1}}{1-\alpha}$ decreases in t. It is easy to ascertain the limiting values of $\gamma(t)$, $g^*(t)$ and $s^*(t)$. Since $\ell(t) \to 0$, we have $r(t) \to \rho$. Therefore, $\gamma(t) \to \rho/\sum_{u=0}^{\infty} p(u)$, or

$$\gamma(t) \to \rho/e \sum_{u=0}^{\infty} P(u)g^* = \rho/eLg^*. \tag{16}$$

In (16) the matrix $L = \sum_{u=0}^{\infty} P(u)$ is simply the matrix of average lifetimes. It follows immediately that

$$\text{(17)} \qquad \begin{aligned} &\text{(i)} \quad g^*(t) \to \frac{\rho g^*}{eLg^*}, \\ &\text{(ii)} \quad s^*(t) \to \frac{\rho Lg^*}{eLg^*}. \end{aligned}$$

The stock constraint in the limit is verified by checking $ALg^* \geqq 0$.

<u>Problem 3</u>: Suppose the system grows at rate $\theta-1$, where $\theta\alpha < 1$. Show that the results of this section continue to hold with $P(u)$, α and $g(t)$ replaced

by $P'(u) = \theta^{-u}P(u)$, $\alpha' = \theta\alpha$ and $g'(t) = \theta^{-t}g(t)$.

Problem 4: Instead of a total size, $es(t) = \rho$, constraint, suppose we have a constraint on some weighted size measure $fs(t) = \rho$ where f is a positive vector. Verify that this will not affect the arguments presented in this section if (12) has a nonnegative solution.

Problem 5: Suppose $P(u) = Q^u$. This implies the model is cross sectional. Show that

$$\text{(i)} \quad \tilde{P}(\alpha) = (I-\alpha Q)^{-1},$$

$$\text{(ii)} \quad \tilde{\ell}(\alpha) = \alpha\tilde{P}(\alpha)Qs(0),$$

$$\text{(iii)} \quad g^*(t) = \frac{g^*w}{eg^*} s^*(t),$$

where w is a row vector, g^*w an $N \times N$ matrix, and $w_i = q_{i0}$ is the fraction of those in class i that leave the system. Note that eg^* is a scalar.

Problem 6: If the objective is to minimize the average cost per period, show how an "approximately" optimal solution can be obtained from the linear program.

$$\begin{aligned} \text{Minimize} \quad & cg, \\ & ALg \geqq 0, \\ & eLg = \rho, \\ & Bg \geqq 0, \\ & g \geqq 0, \end{aligned}$$

where $L = \sum_{u=0}^{\infty} P(u)$, and $c = b + aL$.

Problem 7: Suppose the constraints on $g(t)$ are of the form

$$g(t) - Fz(t) = d, \quad g(t) \geqq 0, \quad z(t) \geqq 0$$

where F is nonnegative. In particular, if F is the identity matrix then we have $g(t) \geqq d$. Show that the results of this section continue to hold with

$$\ell'(t) = \ell(t) + \sum_{j=1}^{t} P(t-u)d,$$

$$g'(t) = g(t) - d.$$

Hint: There always exists a matrix B such that $Bx \geqq 0$ if and only if $x = Fz$ for some $z \geqq 0$. □

We can use the dual of linear program (11) to obtain some idea of the sensitivity of the optimal lower bound and the policy g^* to changes in some of the data. The dual problem is

$$\text{Maximize} \quad \left[\phi \frac{\alpha\rho}{1-\alpha} - e\tilde{\ell}(\alpha)\right] - vA\tilde{\ell}(\alpha)$$

$$\text{subject to} \quad \phi\, e\tilde{P}(\alpha) + vA\tilde{P}(\alpha) + wB \geqq c$$

$$\phi \text{ unrestricted}, \quad v \geqq 0, \quad w \geqq 0.$$

Example 2: Recall that the total lower bound on the optimal cost of (10) is $a\tilde{\ell}(\alpha) + cg^*$ where g^* solves (11). Suppose we wish to change to $\ell'(\alpha) = \ell(\alpha) + \delta\ell$. The change in minimum cost (for $\delta\ell$ small) is

$$(a - \phi e - vA)\delta\ell$$

where ϕ and v are part of an optimal solution to (18).

Problem 8: Suppose a change is made to $a' = a + \delta a$, Show that the change in minimum cost is roughly $\delta a[\tilde{\ell}(\alpha) + \tilde{P}(\alpha)g^*]$.

Problem 9: Continuation of problem 1. Suppose the first row of A is $\lambda f - h$, and that a change $\lambda' = \lambda + \delta\lambda$ is made. Show that the change in the optimal value of (11) is equal to $-v_1 f\tilde{P}(\alpha) g^*$.

Problem 10. Let $P' = \sum_{u=0}^{\infty} u\alpha^{u-1} P(u)$, $\ell' = \sum_{t=1}^{\infty} t\alpha^{f-1} \ell(t)$. Show that the change in optimal value of (11) due to a small change in α is

$$(a - vA - \phi e)(\ell' + P' g^*) + \frac{\phi\rho}{(1-\alpha)^2}\ .$$

□

3. Faculty Promotion Policy; An Example.

This section presents an analysis of faculty promotion policy based on the theory developed in section 2. The intent is for the reader to benefit from the model building and analysis in this section without a detailed understanding of the more technical material in section 2.

Almost all university faculty systems employ some form of tenure policy. Usually a new appointment to the faculty is given a trial period of less than eight years. Before this period is over the appointee must be either granted tenure or dismissed. Individuals who have been granted tenure cannot be dismissed (except for disciplinary reasons) until they reach a mandatory retirement age. The promotion policy of the institution is critical since it is one of the few factors which decision makers can control in determining the structure of the faculty. We first give a simple example to illustrate the relations between promotion policy and i) fraction tenured, ii) new appointments, and iii) time spent in nontenured ranks. This simple example is followed by a more extensive model which illustrates the theory of section 2.

Consider a faculty with two manpower classes, nontenure and tenure. Suppose that each year the institution appoints a total of $g_1 + g_2$ new faculty members, and all these spend 7 years in nontenured positions. Following this seven year period, g_1 of them continue and spend 28 years in tenured positions, whereas g_2 leave having failed to be promoted. Let s_1 and s_2 be the steady state stocks of nontenured and tenured faculty. Then from equation III.5,

$$s_1 = 7g_1 + 7g_2$$

and

$$s_2 = 28g_1 .$$

For example when $g_1 = g_2 = 5$, then $s_1 = 70$ and $s_2 = 140$.

Let the fraction of new appointments that are promoted be θ, so that

$$\theta = g_1/(g_1+g_2).$$

Also assume that the faculty is of fixed size equal to 210. Then the stocks can be written in terms of the promotion fraction θ,

$$s_1 = 210/(1+4\theta)$$

and

$$s_2 = 840\theta/(1+4\theta).$$

From these the fraction of faculty that is tenured is

$$s_2/(s_1+s_2) = 4\theta/(1+4\theta).$$

Figure V.1 shows how the fraction of total faculty that is tenured, varies with the fraction promoted. Clearly as the fraction promoted increases the fraction tenured increases. Suppose we are interested also in how the number of new appointments varies with the fraction promoted. From the above relations it follows that

$$g_1 + g_2 = 30/(1+4\theta).$$

This relation is shown plotted in Figure V.2. Clearly an increase in the fraction promoted leads to a decrease in the number of new appointments.

Let us now fix the promotion fraction θ at 0.5, so that $g_1 = g_2$, keep the total faculty size at 210, and let the time spent in nontenure be a variable ℓ. If the total time spend in the system for those promoted to tenure is 35 years, then

$$s_1 = \ell g_1 + \ell g_2$$

and

$$s_2 = (35-\ell) g_1.$$

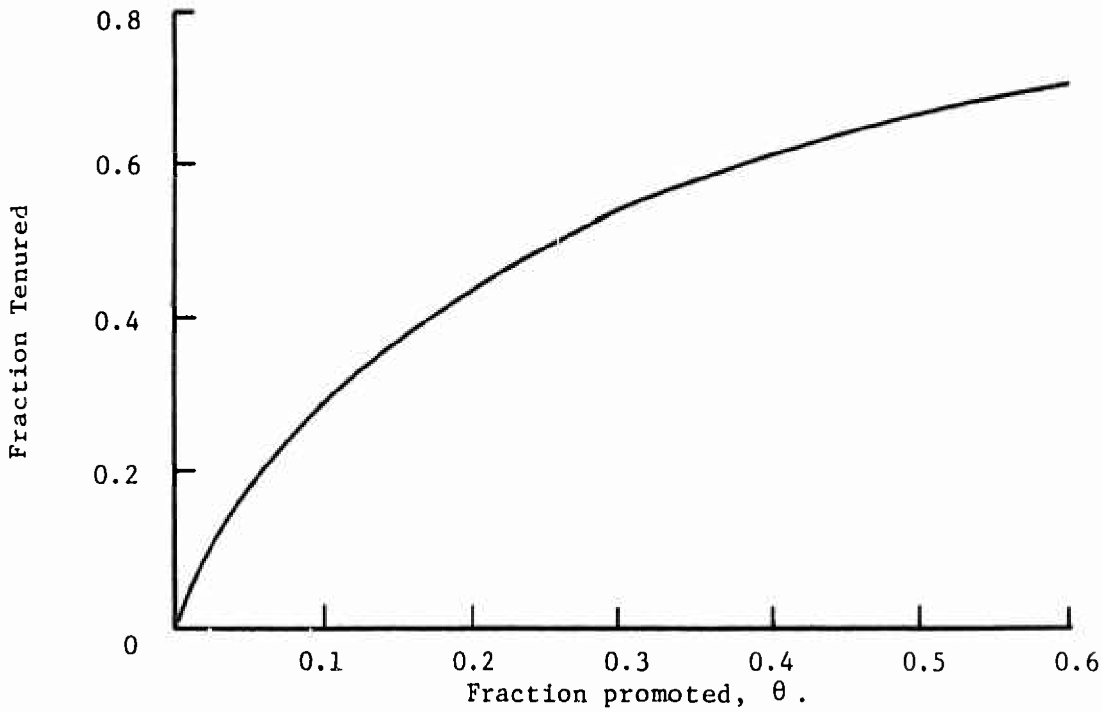

Figure V.1. Fraction tenured vs fraction promoted.

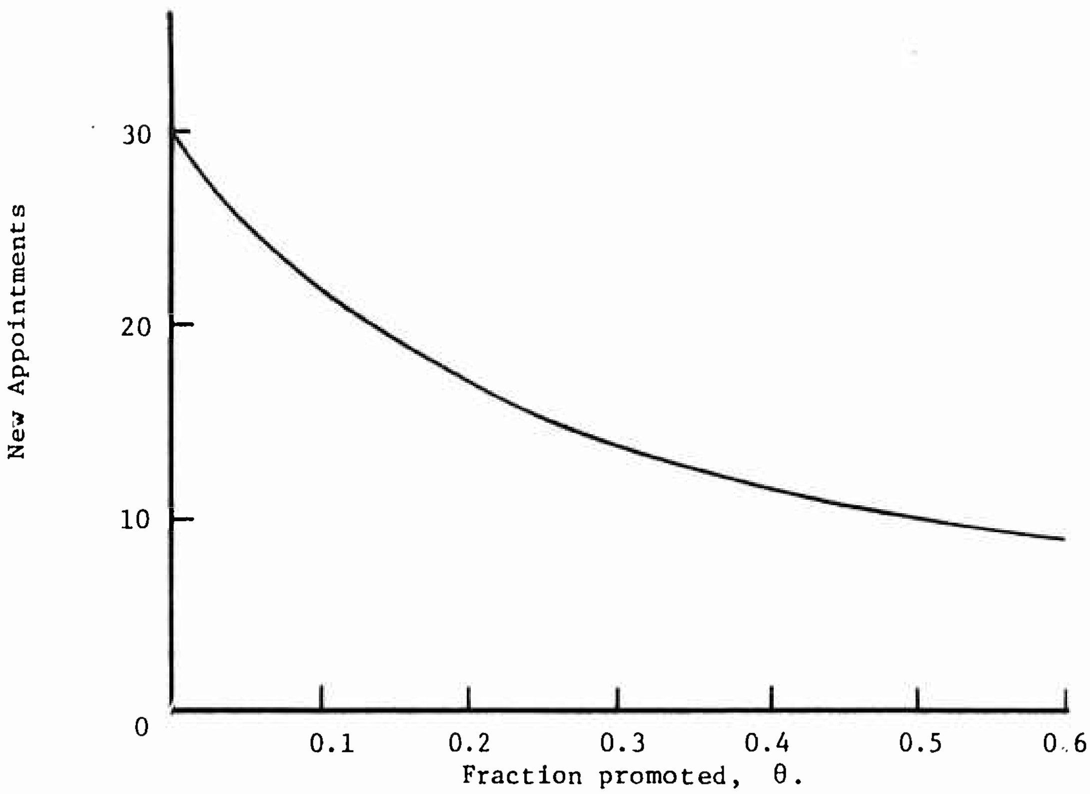

Figure V.2. New appointments vs fraction promoted.

It is easy to see that the steady state fraction of faculty with tenure is $(35-\ell)/(35+\ell)$, and the number of new appointments is $420/(35+\ell)$. Note that a decrease in ℓ from 7 years to 5 years increases the tenure fraction from 0.66 to 0.75. It also increases the new appointments from 10 to 10.5.

Most faculty and administrators would like to have a high rate of new appointments, high promotion rates, short periods in nontenure, and a reasonable fraction of the faculty without tenure. Clearly one must strike a balance among these factors and in a complex system it is not always possible to see exactly where the balance should lie. In the remainder of this section we demonstrate the use of the theory of section 2 in "optimizing" among the various factors.

We build a model with two manpower classes, nontenure and tenure, and fifteen chains (or career paths). The fifteen chains are as follows:

(i) For $k = 1,2,\ldots,7$, a person a chain k is appointed without tenure, spends k years in nontenure ranks, is promoted to tenure and spends a total of between 30 and 40 years in the system. The distribution of total time spent in the system is uniformly distributed between 30 and 40.

(ii) For $k = 8,9,\ldots,14$, a person on chain k is appointed without tenure, spends $(k-7)$ years in nontenure ranks and then leaves.

(iii) A person on chain 15 is appointed with tenure and the time he spends in the system is uniformly distributed between 20 and 30 years.

For completeness the 39 2×15 matrices $P(u)$, $u = 0,1,\ldots,38$ are shown in Table V.1. As usual, blank entries represent zeros. For $u > 38$ $P(u)$ are all zero matrices.

Our object is to formulate a planning problem in the format of (11) in Section 2. The reader should check that A is a matrix for constraints on

Periods of Service	Chain k														
	Nontenure appointments promoted							Nontenure appointments not promoted							Tenure appts.
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	
															1.0
1		1.0	1.0	1.0	1.0	1.0	1.0		1.0	1.0	1.0	1.0	1.0	1.0	
	1.0														1.0
2			1.0	1.0	1.0	1.0	1.0			1.0	1.0	1.0	1.0	1.0	
	1.0	1.0													1.0
3				1.0	1.0	1.0	1.0				1.0	1.0	1.0	1.0	
	1.0	1.0	1.0												1.0
4					1.0	1.0	1.0					1.0	1.0	1.0	
	1.0	1.0	1.0	1.0											1.0
5						1.0	1.0						1.0	1.0	
	1.0	1.0	1.0	1.0	1.0										1.0
6							1.0							1.0	
	1.0	1.0	1.0	1.0	1.0	1.0									1.0
7	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
8	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
9	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
12	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
13	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
14	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
15	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
16	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
17	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
18	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
19	1.0	1.0	1.0	1.0	1.0	1.0	1.0								1.0
20	1.0	1.0	1.0	1.0	1.0	1.0	1.0								0.9
21	1.0	1.0	1.0	1.0	1.0	1.0	1.0								0.8
22	1.0	1.0	1.0	1.0	1.0	1.0	1.0								0.7

Table V.1. The matrices $P(u)$ for the 15-chain faculty example.

23	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.6
24	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.5
25	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.4
26	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.3
27	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.2
28	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1
29	1.0	1.0	1.0	1.0	1.0	1.0	1.0	
30	0.9	0.9	0.9	0.9	0.9	0.9	0.9	
31	0.8	0.8	0.8	0.8	0.8	0.8	0.8	
32	0.7	0.7	0.7	0.7	0.7	0.7	0.7	
33	0.6	0.6	0.6	0.6	0.6	0.6	0.6	
34	0.5	0.5	0.5	0.5	0.5	0.5	0.5	
35	0.4	0.4	0.4	0.4	0.4	0.4	0.4	
36	0.3	0.3	0.3	0.3	0.3	0.3	0.3	
37	0.2	0.2	0.2	0.2	0.2	0.2	0.2	
38	0.1	0.1	0.1	0.1	0.1	0.1	0.1	

Table V.1. (Continued)

stocks, and B is a matrix for constraints on flows. $\tilde{P}(\alpha)$ is determined from the matrices $P(u)$ and the discount factor α which we take to be 0.95. The vector $\tilde{\ell}(\alpha)$ is determined from the legacies as we see below, and ρ is the fixed size of the faculty system.

In our example we place no constraints on stocks other than the one of fixed total faculty size. Thus the constraints with the A matrix drop out of the problem. However, we place several constraints on the system which lead to a matrix B.

(i) Constraint on fraction promoted.

Let us assume that we require that at least a fraction α_1 of new non-tenure appointments get promoted to tenure. Then

$$\left(\sum_{k=1}^{7} g_k\right) / \left(\sum_{k=1}^{14} g_k\right) \geq \alpha_1 .$$

This can be rewritten

$$(1-\alpha_1) \sum_{k=1}^{7} g_k - \alpha_1 \sum_{k=8}^{14} g_k \geq 0 .$$

In the example we assume $\alpha_1 = 0.33$.

(ii) Constraint on tenure appointments.

To allow for flexibility we require that a fraction α_2 of all new appointments be made at the tenure level. Thus

$$g_{15} / \left(\sum_{k=1}^{15} g_k\right) \geq \alpha_2 .$$

This can be rewritten

$$- \alpha_2 \sum_{k=1}^{14} g_k + (1-\alpha_2) g_{15} \geq 0 .$$

In the example we assume $\alpha_2 = 0.025$.

(iii) Constraints on average nontenure lifetimes.

Low average times in nontenure ranks make a university attractive to prospective appointees who have high expectations of being promoted. For those not promoted the university could keep a faculty member without tenure for the full seven years and be very economical. But such a policy is somewhat unfair to the faculty member who stays with no hope of promotion and also prevents a new appointment into his position. Thus we set upper bounds on the average time spent without tenure.

Let τ_1 and τ_2 be upper bounds on the average times spent in nontenure ranks for those eventually promoted and those not promoted respectively. Then

$$\left(\sum_{k=1}^{7} kg_k\right) \Big/ \left(\sum_{k=1}^{7} g_k\right) \leq \tau_1,$$

and

$$\left(\sum_{k=8}^{14} (k-7)g_k\right) \Big/ \left(\sum_{k=8}^{14} g_k\right) \leq \tau_2.$$

These can be rewritten

$$\sum_{k=1}^{7} (\tau_1 - k)g_k \geq 0,$$

and

$$\sum_{k=8}^{14} (\tau_2 - k + 7)g_k \geq 0.$$

In the example we assume $\tau_1 = 5.5$ years, and $\tau_2 = 4.5$ years.

(iv) Constraint on long range tenure fraction.

In the long run we require that the fraction of faculty in tenure ranks be no more than α_3. Let ℓ_{1k} and ℓ_{2k} be the expected number of years spent in nontenure and tenure ranks for chain k. Clearly, if L is the 2×15 matrix of the numbers, $L = \sum_{n=0}^{\infty} P(u)$. This constraint can be written

$$\sum_{k=1}^{15} \ell_{2k} g_k \Big/ \sum_{k=1}^{15} (\ell_{1k}+\ell_{2k}) g_k \le \alpha_3,$$

or

$$\sum_{k=1}^{15} (\alpha_3 \ell_{1k} - (1-\alpha_3)\ell_{2k}) g_k \ge 0.$$

In the example we assume $\alpha_3 = 0.70$.

The above four sets of constraints form the $Bg \ge 0$ constraints of (11). To complete the model formulation we need the legacies $\ell(t)$ and the costs a and b. The costs on input flows, b, are assumed to be zero. The costs (in units of \$1000) on stocks, per period, are

$$a = (14.5,\ 28).$$

$\tilde{P}(\alpha)$ is calculated from the $P(u)$ matrices in Table V.1 and $\alpha = 0.95$ and is shown together with $P(\alpha)$ in Table V.2. This gives

$$c = a\tilde{P}(\alpha) + b,$$

a 15-component vector also shown in Table V.2.

The legacies $\ell(t)$ are shown in Table V.3 (the numbers in the last column are explained later). These give the numbers in each class in each future period if no new inputs were made into the system. From this data

$$\tilde{\ell}(\alpha) = \sum_{t=1}^{\infty} \alpha^t \ell(t) = [656,\ 6883],$$

and $e\tilde{\ell}(\alpha) = 7539$. The total faculty size ρ is taken to be 1000, so that the right-hand-side coefficient (see (11))

$$[\alpha\rho/(1-\alpha)] - e\tilde{\ell}(\alpha) = 11{,}461.$$

The linear program (11) can now be written out, with 15 variables $g_1, g_2, \ldots, g_{15}$, and 6 constraints. The whole program is shown in Table V.4. The last row gives the optimal solution g^*, and the minimum total future

Chain k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\tilde{P}(\alpha)$	1.00	1.95	2.85	3.71	4.52	5.30	6.03	1.00	1.95	2.85	3.71	4.52	5.30	6.03	0.00
	15.55	14.60	13.70	12.84	12.03	11.25	10.52	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$e\tilde{P}(\alpha)$	16.55	16.55	16.55	16.55	16.55	16.55	16.55	1.00	1.95	2.85	3.71	4.52	5.30	6.03	14.25
c	450.0	437.2	425.0	413.5	402.5	392.0	382.1	14.5	28.3	41.4	53.8	65.6	76.8	87.5	398.9

Table V.2. Cost vector c and constraint vector $e\tilde{P}(\alpha)$ for the 15-chain example.

	Legacies $\ell(t)$		
t	Nontenure	Tenure	$\rho-e\ell(t)$
1	264	623	113
2	196	621	183
3	135	617	248
4	84	611	305
5	43	603	354
6	15	591	394
7		574	426
8		553	447
9		531	469
10		510	490
11		489	511
12		467	533
13		446	554
14		424	576
15		403	597
16		381	619
17		360	640
18		339	661
19		317	683
20		296	704
21		274	726
22		253	747
23		232	768
24		212	788
25		191	809
26		170	830
27		150	850
28		130	870
29		110	890
30		90	910
31		72	928
32		56	944
33		42	958
34		30	970
35		20	980
36		12	988
37		6	994
38		2	998
39		0	1000

Table V.3. Legacies in nontenure and tenure ranks.

	Variable																
	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}		Right Hand Side
$e\tilde{P}(\alpha)$	16.55	16.55	16.55	16.55	16.55	16.55	16.55	1.00	1.95	2.85	3.71	4.52	5.30	6.03	14.25	=	11,461
	0.67	0.67	0.67	0.67	0.67	0.67	0.67	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	0.0	≥	0
	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	0.975	≥	0
B	4.5	3.5	2.5	1.5	0.5	-0.5	-1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	≥	0
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.5	2.5	1.5	0.5	-0.5	-1.5	-2.5	0.0	≥	0
	-9.35	-8.35	-7.35	-6.35	-5.35	-4.35	-3.35	0.7	1.4	2.1	2.8	3.5	4.2	4.9	-7.35	≥	0
Object. Funct.	438.1	422.0	407.0	396.0	389.0	388.7	92.0	14.5	28.3	41.4	53.8	65.6	76.8	87.5	399.0	Minimize	
Opt. Soln.	0	0	0	0	220	220	0	0	0	0	447	447	0	0	34		242,088

Table V.4. Linear Program (11) for the 15 chain example.

expected cost. It now remains to calculate the numbers $\{\gamma(t)\}$ using (12) of section 2 in order to obtain the optimal hiring and promotion policy from g^*.

Recall that the right-hand-side of the t-th equation in (11) is $r(t) = \rho - e\ell(t)$. These numbers are shown in the last column of Table V.3. The coefficients $\{p(u)\}$ are given by $p(u) = eP(u)g^*$. The reader can check that the first six elements are given by

u	0	1	2	3	4	5
p(u)	1368	1368	1368	1368	921	474

From equations (11) we obtain

t	1	2	3	4	5	6	∞
γ(t)	0.083	0.051	0.047	0.042	.063	0.073	0.048

These lead to the optimal hiring and promotion policy for the first six periods shown in Table V.5, and the steady state policy. All numbers have been rounded to the nearest integer.

We now interpret the results in Table V.5. First we see that we have a stationary policy with respect to time to promotion and time to withdrawal of unpromoted faculty. This is a highly desirable feature of the model which arises from the form of the solution $g^*(t) = \gamma(t)g^*$, since it leads to smooth operating policies. With the costs and constraint parameters given, the optimal course of action is to promote faculty after 5 and 6 years in nontenure ranks, and to require them to leave after 4 and 5 years if they are not to be promoted. The 33% promotion constraint is binding, as is the 2.5% constraint on appointments with tenure.

Period	Nontenure appointments, promoted							Nontenure appointments, not promoted							Tenure Appointments	Total
t	1	2	3	4	5	6	7	8	9	10	11	12	13	14		
1					18	18					37	37			3	113
2					11	11					23	23			2	70
3					10	10					21	21			2	64
4					9	9					19	19			1	57
5					14	14					28	28			2	86
6					16	16					33	33			2	100
∞					10	10					21	21			2	64

Table V.5. Optimal hiring and promotion policy for periods 1-6, and steady state.

The results in Table V.5 are from one run of the model under the various constraints mentioned. What a decision maker can do is to try changing various constraints and see what effect these changes have on the optimal solution. We stress again that the answers presented in Table V.5 are not to be interpreted as a "take-it-or-leave-it" solution. This small linear program is so easily and quickly solved that it is best done at a computer terminal in interactive mode so that various policies which affect the constraints can be changed. Recall that we required that no more than 70% of the faculty in any period be in tenure positions. If this is reduced to 65% the program (11) has no feasible solution. Thus important information can be obtained by trying different values of the parameters in the model.

4. Transient Problems.

This section shows how to combine an initial planning phase with the long run planning problem described in Section 2. The initial decisions to be made by an organization are of far greater significance than the planned future decisions, since one can always revise future plans. We do not have the luxury of revising today's decision, but have to make the decision and live with the results. A useful planning tool, therefore, concentrates on the initial decisions, since they have an immediate impact and are more likely to be precisely implemented. At the same time a planning model should consider the long range implications of present decisions, so it is important to bring the long run or steady state analysis into the model in some way. We can do this by combining an initial T stage planning problem with infinite stage problems of the type described in Section 2. The result is a linear program with the format of a $T+1$ stage planning model. As we have pointed out in previous sections and the introduction to this chapter, the explicit decision variables and objective function of the linear programming model are not of overriding significance. Much of the data that determine the flow process, the level of constraints, and the costs associated with the manpower stocks and flows result from important policy decisions. If the linear programming model is flexible, it is possible to calculate the impact on the system of changes in these policy decisions. In this way the planning model becomes a useful device for simulating the impact of proposed policies or even for designing new policies.

The initial planning phase consists of the selection of input flow vectors $g(1),\ldots,g(T)$. These are constrained by the linear inequalities:

$$\sum_{t=1}^{T} F(t)g(t) \geqq d, \qquad g(t) \geqq 0, \qquad 1 \leqq t \leqq T. \tag{19}$$

We are vague about the specific interpretation of these constraints since they can be any constraints that we desire with the given mathematical form. One particularly interesting set of constraints is the first T constraints of the long run problem. This set of constraints would result in the rigid enforcement of the constraint $As(t) \geqq 0$ in periods 1 through T. Another simple example is a set of constraints which allow alternate growth paths for an organization that is growing to a specified steady state size. The flow vectors $g(1), g(2), \ldots, g(T)$ determine a legacy in future periods $T+1, T+2, \ldots$ Let $\ell(T,t)$ be the legacy in period t, for $t \geq T$. Then

$$\ell(T,t) = \ell(t) + \sum_{j=1}^{T} P(t-j)g(j), \tag{20}$$

where $\ell(t)$ is simply the legacy due to decisions before time zero. The stock of manpower at time $t \geq T+1$ is given by

$$s(t) = \ell(T,t) + \sum_{j=T+1}^{t} P(t-j)g(j). \tag{21}$$

Let ρ be the stipulated size of the organization at time T and $\theta - 1$ the growth rate in periods $T+1$, $T+2$, etc. The constraints on $g(t)$ for $t \geqq T+1$ are:

$$\begin{gathered} A \sum_{j=T+1}^{t} P(t-j)g(j) \geqq -A\ell(T,t), \\ e \sum_{j=T+1}^{t} P(t-j)g(j) = \rho - e\ell(T,t), \\ Bg(t) \geqq 0, \quad g(t) \geqq 0. \end{gathered} \tag{22}$$

The present value (at time 0) of the costs incurred from time $T+1$ onwards is given by

$$(23)\quad \sum_{t=T+1}^{\infty} \alpha^t[as(t)+bg(t)] = \sum_{t=T+1}^{\infty} \alpha^t a\ell(t) + \sum_{t=1}^{T} \alpha^t a\left(\sum_{n=T+1-t}^{\infty} \alpha^u P(u)\right)g(t) + \sum_{t=T+1}^{\infty} \alpha^t cg(t)$$

where $c = b + a\sum_{u=0}^{\infty}\alpha^u P(u)$. Notice that only the last term of (23) depends on decisions taken after time T.

Problem 11: Verify (23) □

From the results of Section 2, we know that for any value of $g(1), g(2), \ldots, g(T)$, we can find an approximately optimal operating policy from time $T+1$ onwards by solving a linear program. The cost contribution of $g(t)$ for $t \geqq T+1$ is

$$(24)\quad \sum_{t=T+1}^{\infty} \alpha^t cg(t) = cg \quad \text{where} \quad g = \sum_{t+T+1}^{\infty} \alpha^t g(t).$$

By using (22) and (24) we see that the linear program for calculating an approximately optimal solution from time $T+1$ onward is

(25) Minimize cg

subject to $A\tilde{P}(\alpha)g \geqq -A\tilde{\ell}(T,\alpha)$

$e\tilde{P}(\alpha)g = (\rho/1-\alpha) - e\tilde{\ell}(T,\alpha)$,

$Bg \geqq 0, \quad g \geqq 0$,

where

$$\text{(i)}\quad g = \sum_{t+T+1}^{\infty} \alpha^t g(t),$$

and

$$\text{(ii)}\quad \tilde{\ell}(T,\alpha) = \sum_{t+T+1}^{\infty} \alpha^t \ell(T,t).$$

Problem 12:

Show that

$$\tilde{\ell}(T,\alpha) = \sum_{t=T+1}^{\infty} \alpha^t \ell(t) + \sum_{j=1}^{T} \alpha^j\left[\sum_{u=T+1-j}^{\infty} \alpha^u P(u)\right]g(j).$$

Hint: Use equation (20). □

Let $c(1), c(2), \ldots, c(T)$ be the costs associated with selecting $g(1), \ldots, g(T)$. To avoid double counting we assume the $c(t)$ refers to costs actually incurred in periods t, for $1 \leq t \leq T$.

<u>Example 3</u>: Let $a(t)$ and $b(t)$ be the respective costs of stocks and flows in period t. The total costs incurred in periods 1 through T are:

$$\sum_{t=1}^{T} \alpha^t [a(t)s(t)+b(t)g(t)]$$
$$= \sum_{t=1}^{T} \alpha^t a(t)\ell(t) + \sum_{t=1}^{T} \alpha^t [b(t) + \sum_{j=t}^{T} \alpha^{t-j} a(j)P(t-j)]g(t).$$

The cost incurred in period t is

$$c(t) = b(t) + \sum_{u=0}^{T-t} \alpha^u a(t+u)P(u). \tag{27}$$ □

The linear program (25) is used to select an optimal longrun policy <u>given</u> $g(1), \ldots, g(T)$. We can put the initial constraints on $g(1)$ through $g(T)$ together with (25) and simultaneously choose $g(1), \ldots, g(T)$ and g. The total cost of any such plan is, from (23) and (27),

$$\sum_{t=1}^{\infty} \alpha^t a(t)\ell(t) + \sum_{t=1}^{T} \alpha^t [b(t) + \sum_{u=0}^{T-t} \alpha^u a(t+u)P(u)]g(t) + cg. \tag{28}$$

The entire linear program becomes

$$\text{Minimize} \quad \sum_{t=1}^{T} \alpha^t c(t)g(t) + cg,$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{t=1}^{T} F(t)g(t) \geq d, \\
& A \sum_{t=1}^{T} \alpha^t \left[\sum_{u=T+1-t}^{\infty} \alpha^u P(u) \right] g(t) + A\tilde{P}(\alpha)g \geqq -A \sum_{t=T+1}^{\infty} \alpha^t \ell(t), \\
& e \sum_{t=1}^{T} \alpha^t \left[\sum_{u=T+1-t}^{\infty} \alpha^u P(u) \right] g(t) + e\tilde{P}(\alpha)g = \rho - e \sum_{t=T+1}^{\infty} \alpha^t \ell(t), \\
& Bg \geq 0, \\
& g \geq 0 \\
& g(t) \geq 0, \qquad t = 1,2,\ldots,T.
\end{aligned} \tag{29}$$

This rather formidable linear program solves the combined transient-steady state problem. In the next section we demonstrate some specific models that exploit the ideas presented in this section.

5. Optimization in a One Class, One Chain Model.

This section examines some optimization problems based on the one class, one chain longitudinal model of Section 3, Chapter III. The simplicity of this model allows us to obtain sharper and more interesting results than are possible with the general longitudinal model. The usefulness of the analysis and results lies in the fact that a one class, one chain model is the result of aggregating over classes and chains. In complex planning situations it is frequently possible to devise a classification scheme to allow for nearly independent operation of each manpower class. An understanding of simple, although imperfect, models of the manpower system is an indispensible aid in the construction of more sophisticated models.

The first problem considered is that of meeting given manpower requirements of minimum cost. A related problem of maximizing a measure of effectiveness subject to budget constraints is discussed later in the section. Both problems are formulated as infinite horizon linear programs. The exact form of the optimum policy is determined and it is shown how this optimal policy is well approximated by the solution of a finite horizon (usually quite small) linear program. The theory is presented in its simplest form. Extensions are left as problems for the interested reader.

In the basic one class, one chain model $g(t)$ gives the number of new input into the system in period t, and $p(u)$ gives the fraction of those who remain in the system at least u periods. Thus if $s(t)$ are the stocks at time t,

$$(30) \qquad s(t) = \ell(t) + \sum_{j=1}^{t} p(t-j)g(j), \qquad t \geq 1,$$

where $\ell(t)$ is the "legacy" at time t due to accessions prior to time zero. Now let $c(u)$ be the cost of support for an individual who has length of completed service equal to u, and let α be a discount rate applied to future costs. The present value (at time zero) of the manpower costs incurred in period t is therefore

$$\alpha^t \sum_{u=0}^{\infty} c(u)p(u)g(t-u).$$

This expression can be written as

$$\alpha^t \left[\sum_{j=1}^{t} c(t-j)p(t-j)g(j) + h(t) \right],$$

where $h(t) = \sum_{j\leq 0} c(t-j)p(t-j)g(j)$. The second term in this expression, $h(t)$, is a "cost legacy" due to accessions prior to time zero. Let

$$\tilde{c}(\alpha) = \sum_{u=0}^{\infty} \alpha^u c(u)p(u),$$

which can be interpreted as the expected total discounted cost of an accession. Then the present value of all future costs is given by

$$\tilde{c}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t) + \sum_{t=1}^{\infty} \alpha^t h(t). \tag{31}$$

<u>Problem 13</u>: To obtain (31) one needs the result

$$\sum_{t=1}^{\infty} \alpha^t \sum_{j=1}^{T} c(t-j)p(t-j)g(j) = \tilde{c}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t).$$

Verify this result.

<u>Problem 14</u>: Let $r(u) = p(u-1) - p(u)$, $u \geq 1$. Then $r(u)$ is the probability that an individual stays in the system exactly u periods. Also let $d(u) = \sum_{j=0}^{u-1} \alpha^j c(j)$, so that $d(u)$ is the present value of the cost of a person who stays in the system u periods. Then $\sum_{u=1}^{\infty} r(u)d(u)$ is the expected total discounted cost of an accession. Show that

$$\sum_{u=1}^{\infty} r(u)d(u) = \sum_{u=0}^{\infty} \alpha^u c(u)p(u).$$

Example 4: For a system where the maximum number of periods in the system, M, is 5 (so that $p(u) = 0,\ u \geq 6$), let the lifetime distribution and costs be given by

u	0	1	2	3	4	5
p(u)	1.0	1.0	0.9	0.8	0.5	0.2
c(u)	15.0	6.0	8.0	11.0	14.0	18.0

.

Let the discount factor α be 0.9, and past accessions be given by

t	-4	-3	-2	-1	0
g(t)	400	600	800	1000	1000

.

Note that for periods prior to -4 all accession have left the system by time zero. For the above data the legacy and cost legacy are given by

t	1	2	3	4	5
$\ell(t)$	2920	2220	1460	700	200
h(t)	25880	23760	18680	10600	3600

.

Problem 15: Based on example 4.

If $\alpha = 0.9$, what is the value of $\tilde{c}(\alpha)$?

□

Let $a(t)$ be the manpower requirement at time t. The value $a(t)$ is interpreted as a minimum requirement, so that the stocks at time t must be at least as large as $a(t)$. Thus from (30),

$$\ell(t) + \sum_{j=1}^{t} p(t-j)g(j) \geq a(t), \qquad t \geq 1.$$

Now let $a(t) - \ell(t) = b(t)$, so that $b(t)$ is the net requirement at time t (actual minus legacy) which must be obtained from accessions in periods $1,2,\ldots,t$. The minimum requirement constraint thus becomes

$$\sum_{j=1}^{t} p(t-j)g(j) \geq b(t).$$

The problem of minimizing the cost of meeting a schedule of future (net) requirements is to find future accessions $\{g(t)\}$ in order to

$$\text{(33)} \qquad \text{Minimize} \quad \tilde{c}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t)$$

$$\text{subject to} \quad \sum_{j=1}^{t} p(t-j)g(j) \geq b(t)$$

$$g(t) \geq 0, \qquad t = 1,2,\ldots \ .$$

The reader should notice that as long as $\tilde{c}(\alpha)$ is positive (certainly the usual case!) its value will not affect the optimal accessions policy. Although it affects the value of the objective function in (33), since it is simply a multiplicative constant it has no effect on the optimal value of the variables. Thus (33) can be solved without any knowledge of the costs $c(u)$. The only assumption required is that these costs depend only on a person's length of service u, and not on when the person entered the system.

Problem 16: Suppose that there is a lower bound $x(t)$ on accessions in period t. Show that the linear program (33) is the correct problem formulation if $g(t)$ is interpreted to be the number of accessions above the lower bound and

$$b(t) = a(t) - \ell(t) - \sum_{j=1}^{t} p(t-j)x(j).$$

What constant must be added to the variables obtained from (33) in order to calculate the total expected future cost?

Problem 17: Suppose that $a(t)$ measures the demand for "effective" manpower, and let $e(u)$ be the "effectiveness" of an individual who has u periods of completed service. If $b(t)$ is the net demand for effective manpower, show how to compute an "effectiveness" legacy, and show that the constraints in (33) become

$$\sum_{j=1}^{t} e(t-j)p(t-j)g(j) \geq b(t).$$

Problem 18: Continuation of problem 15, based on example 4.

Let the effective manpower demands, $a(t)$, be given by

t	1	2	3	4	5	6
a(t)	3120	2300	2150	2000	2000	2000

Calculate a feasible solution of (33) for 6 periods. What should be the value to $g(t)$ for large t?

□

Suppose that $p(0) > 0$. This is a trivial assumption which says that not everyone leaves the system the instant they join it! Now define $g^+(t)$ for each $t \geq 1$ by

$$g^+(t) = \text{Max}\{0,[b(t) - \sum_{j=1}^{t-1} p(t-j)g^+(j)]/p(0)\}. \tag{34}$$

The reader should verify that the $\{g^+(t)\}$ forms a feasible solution to (33). Thus (33) is always feasible and we show below that the feasible solution given by (34) is often optimal.

Problem 19: Continuation of problem 18.

Calculate $g^+(t)$ for six periods.

□

The linear program (33) has an infinite number of variables which must satisfy an infinite number of constraints. In general the optimal solution of such problems is impossible to determine. However, the special structure of the manpower flow model allows us to say a great deal about the optimal solution, and indeed allows us to demonstrate that, under certain conditions on the demands, $\{g^+(t)\}$ defined by (34) gives the optimal solution. First we present the dual linear program to (33),

$$\text{Maximize} \quad \sum_{t=1}^{\infty} v(t)b(t),$$

$$\text{(35)} \qquad \text{subject to} \quad \sum_{j=0}^{\infty} v(t+j)p(j) \le \alpha^t \tilde{c}(\alpha),$$

$$v(t) \ge 0, \qquad t = 1,2,\ldots \ .$$

For each $t \ge 1$ define

$$\text{(36)} \qquad v^*(t) = \alpha^t \tilde{c}(\alpha)/\tilde{p}(\alpha),$$

where $$\tilde{p}(\alpha) = \sum_{u=0}^{\infty} \alpha^u p(u).$$

The reader should show that $v^*(t)$ is both feasible and optimal in (35).

Problem 20: Continuation of problem 19.

Calculate $\tilde{c}(\alpha)/\tilde{p}(\alpha)$ for the example, and write out $v^*(t)$, $t = 1,2,3,4,5$.

Problem 21: Continuation of problem 17.

Using the measure of effectiveness $e(t)$ introduced in problem 17, write out the dual linear program which replaces (35) and its optimal solution which replaces (36).

□

In order to investigate further the optimal solution to (33) and use duality theory from linear programming, the following basic assumptions are made:

(37)
(i) $0 < \alpha < 1$.

(ii) $\tilde{c}(\alpha) > 0$.

(iii) $p(0) > 0$, and $p(u) \geq 0$ for all $u \geq 1$.

(iv) $\tilde{b}(\alpha) = \sum_{t=1}^{\infty} \alpha^t b(t) < \infty$.

(v) $b(t) \geq 0$ for all t larger than some T.

In our simple model (33) these assumptions almost trivially hold. Number (iv) says that future net requirements cannot grow too rapidly, whereas number (v) says that, although one may start with negative net requirements (a short-range surplus exists), the net requirements eventually turn positive and remain positive. In the more general case where effectiveness is introduced these assumptions become more restrictive.

<u>Problem 22</u>: Continuation of problems 17 and 21.

For the effectiveness model the assumption corresponding to (37)-(iii) would be $a(0)p(0) > 0$ and $a(u)p(u) \geq 0$, $u \geq 1$. Present an example with $a(u) < 0$ for some u.

□

Under the assumptions in (37) a great deal can be said about the optimal solutions to (33) and (35), and the main result is stated as a theorem.

<u>Theorem 1</u>: Under assumptions (37) the dual linear programs (33) and (35) both have optimal solutions. If these are denoted $\{g^*(t)\}$ and $\{v^*(t)\}$ then the equality

$$\sum_{t=1}^{\infty} v^*(t)b(t) = \tilde{c}(\alpha) \sum_{t=1}^{\infty} \alpha^t g^*(t)$$

also holds.

The proof of this theorem is exceedingly technical and can be found in references given in the "Notes and Comments" section at the end of the chapter. The theorem can be used to show that, if $\{g(t)\}$ and $\{v(t)\}$ are any feasible solutions to (33) and (35) respectively, then they are optimal if and only if

$$\text{(38)}\qquad \begin{aligned} &\text{(i)} \qquad v(t)\left[\sum_{j=1}^{t} p(t-j)g(j) - b(t)\right] = 0, \\ &\text{(ii)} \qquad g(t)\left[\sum_{j=0}^{\infty} v(t+j)p(j) - \alpha^{t}\tilde{c}(\alpha)\right] = 0, \end{aligned}$$

both hold for all $t \geqq 1$.

Problem 23: Using theorem 1 show that the complementary slackness conditions (38) must hold if $\{g(t)\}$ and $\{v(t)\}$ are optimal solutions of (33) and (35).

Problem 24: Continuation of problems 17, 21, 22.

For the example using effective manpower write down the complimentary slackness conditions which replace (38).

□

Theorem 1 and the result (38) are now used to show that under certain conditions the simple feasible solution in (34), $\{g^{+}(t)\}$, is also optimal.

Consider first the accessions required to exactly meet future requirements. Denoting these by $\{g^{\times}(t)\}$ they are the unique solution to the equations

$$\text{(39)}\qquad \sum_{j=1}^{t} p(t-j)g^{\times}(j) = b(t), \qquad t \geq 1.$$

If $g^{\times}(t) \geq 0$ for all $t \geq 1$ then this solution is feasible in (33). Also it is clear that $g^{\times}(t) = g^{+}(t)$, $t \geq 1$. Taking the $\{v^{*}(t)\}$ defined by (36) the reader can see that $\{g^{\times}(t)\}$ and $\{v^{*}(t)\}$ satisfy (38). Thus the simple solution to (39) is optimal whenever it is non-negative.

It is generally impossible to verify whether or not the solution of (39) is non-negative. However, under certain conditions on the net requirements,

$b(t)$, and the flow fractions, $p(u)$, we show that the solution of (39) is non-negative, and hence is optimal.

Under the policy implied by (39) the input flow in period $t+1$ is $g^{\times}(t+1)$. The amount of this which is available to meet requirements at time $t+1$ is $p(0)g^{\times}(t+1)$. To exactly meet requirements this inflow must be equal to any change in net requirements in the period, which is $b(t+1)-b(t)$, plus any losses during the period, which amount to $\sum_{j=1}^{t} [p(t+1-j) - p(t-j)]g^{\times}(j)$. Therefore

$$(40) \qquad p(0)g^{\times}(t+1) = b(t+1) - b(t) + \sum_{j=1}^{t} [p(t+1-j) - p(t-j)]g^{\times}(j).$$

After some rearrangement of terms and with the aid of (39) this can be written as

$$(41) \qquad p(0)g^{\times}(t+1) = \sum_{j=1}^{t} \left[\frac{b(t+1}{b(t)} - \frac{p(t+1-j)}{p(t-j)}\right] p(t-j)g^{\times}(j).$$

Using (41) one can prove

Theorem 2: Under the assumptions in (37), if

$$(42) \qquad \frac{b(t+1)}{b(t)} \geq \frac{p(j)}{p(j-1)} \quad \text{for} \quad 1 \leq j \leq t \quad \text{and} \quad t \geq 1,$$

then the solution of (39) is non-negative and is thus an optimal solution to (33).

Problem 25: Prove theorem 2.

Problem 26: Continuation of problem 17.

How do the conditions in theorem 2 change in the model using manpower effectiveness?

Problem 27: Show that

a) if $p(u) \geq p(u+1)$ for all u, then $\ell(t) \geq \ell(t+1)$ for all t,

b) if in addition $a(t) \leq a(t+1)$ for all $t \geq 1$, then (42) holds.

Problem 28: Continuation of problems 15 and 18.

Does (42) hold for this numerical example?

Problem 29: Continuation of problem 15.

For what values of $a(t)$ is the equality solution optimal?

□

We have shown conditions under which $\{g^{\times}(t)\}$, which solves (39), is non-negative and is thus optimal in (33). We now show that $\{g^{+}(t)\}$ which satisfies (34) is optimal under more general conditions. This is done in

Theorem 3: Under assumptions (37), if in addition

$$p(u) > \alpha p(u+1) \tag{43}$$

for all $u \geq 0$, then $\{g^{+}(t)\}$ which satisfies (34) is an optimal solution of (33).

Proof: Let $\{g^*(t)\}$ and $\{v^*(t)\}$ be optimal solutions of (33) and (35) respectively, and assume for some T that

$$g^*(T) > \mathrm{Max}\{0, [b(t) - \sum_{j=1}^{T-1} p(T-j)g^*(j)]/p(0)\}. \tag{44}$$

From (44) and (38) it follows that

$$\sum_{j=0}^{\infty} v^*(T+j)p(j) = \alpha^T \tilde{c}(\alpha) \tag{45}$$

and

$$v^*(T) = 0.$$

Combining these results gives

$$\sum_{j=1}^{\infty} v^*(T+j)p(j) = \alpha^T \tilde{c}(\alpha).$$

But since $\{v^*(t)\}$ is feasible in (35)

$$\sum_{j=0}^{\infty} v^*(T+1+j)p(j) \leqq \alpha^{T+1}\tilde{c}(\alpha).$$

Multiplying the first of these two relations by $-\alpha$ and adding to the second gives

$$\sum_{j=1}^{\infty} v^*(T+j)[p(j-1) - \alpha p(j)] \leqq 0.$$

The assumption of the theorem, (43), implies that all the terms in square brackets are positive, which implies that since the $\{v^*(t)\}$ are non-negative they must all be zero. However, since if (37) holds, $\alpha^T\tilde{c}(\alpha) > 0$, (45) implies that $v^*(T) > 0$ for some T. This contradiction implies that (44) cannot hold and the theorem is proved.

Example 5: Continuation of example 4 and problem 15.

For the data given the optimal solution to (33), for $1 \leq t \leq 6$ is

t	1	2	3	4	5	6
$g^+(t)$	200	0	510	630	611	374

Problem 30: Continuation of Problem 17.

How must the restriction (43) in theorem 3 be modified for the effectiveness model? Given the data in example 4 and problem 15 what values of $a(u)$ will satisfy this modified inequality?

□

It is often not possible to determine if the equality solution $\{g^\times(t)\}$ is non-negative. In situations which involve relatively large decreases in requirements in the first few periods together with a large legacy, some elements of the solution are often negative. In these cases we must find some procedure for either solving or approximating the solution of the infinite horizon problem.

We briefly describe three methods for calculating approximately optimal solutions to the infinite horizon optimization problem (33). Each of the three methods is based on a partition of the original infinite problem into a T period finite problem followed by an infinite problem that commences at time $T+1$. The hope is that the system will settle down enough so that the problem starting at time $T+1$ will have a non-negative equality solution regardless of the choice of $g(t)$, $1 \leqq t \leqq T$.

The first method simply ignores the decisions and constraints for time $T+1$ onwards. This procedure is quite simple, but it can lead to optimal programs that save in periods 1 through T by presenting difficult initial conditions for the second problem that commences at time $T+1$. Since the problem that starts at time $T+1$ is not explicitly considered in the objective, there is no penalty to deter this type of behavior.

The second method assumes $p(u) = 0$ for $u > M$, and attempts to provide a smooth transition to equilibrium by fixing accessions at their equilibrium value for periods $T+1$ onward. The assumption is that $b(t) = b$ for $t \geq T$, and that $g(t) = b/\sum_{j=0}^{u} p(j)$ for all $t > T$. Thus the accessions in past periods and periods $T+1, T+2, \ldots$ are all known. We must determine the accessions in periods 1 through T in order to satisfy the lower bound requirement in the first $T+M$ time periods. This leads to a linear program with $T+M$ inequality constraints and T non-negative variables $g(t)$; $1 \leqq t \leqq T$. The dual of this linear program has T inequality constraints and $T+M$ non-negative variables and is easier to solve. Unfortunately this truncation procedure has not been effective in numerical examples we have solved. We frequently obtain relatively low values of $g(T-2), g(T-1)$, etc., and a relatively large value of $g(T)$. In effect, the program satisfies the boundary restriction by making a last period correction. This behavior is contrary to the smooth transition to equilibrium that the model was designed to produce.

The third method for approximating solutions is based on the transient analysis done in Section 4. We derive the problem to be solved in a different and interesting way. Consider the dual problem (35), and fix the dual variables $v(t)$ for $t \geqq t+1$ at values $\alpha^t \tilde{c}(\alpha)/\tilde{p}(\alpha)$. The dual becomes

$$\text{Maximize} \quad \sum_{t=1}^{T} v(t)b(t) + \frac{\tilde{c}(\alpha)}{\tilde{p}(\alpha)} \sum_{t=T+1}^{\infty} \alpha^t b(t)$$

$$\text{(46)} \qquad \text{subject to} \quad \sum_{j=0}^{T-t} v(t+j)p(j) \leqq \alpha^t \tilde{c}(\alpha) - \sum_{j=T+1-t}^{\infty} \alpha^{t+j} \frac{\tilde{c}(\alpha)}{\tilde{p}(\alpha)} p(j),$$

$$v(t) \geqq 0, \qquad t = 1,2,\ldots,T.$$

Problem 31: Show that

$$\alpha^t \tilde{c}(\alpha) - \sum_{j=T+1-t}^{\infty} \alpha^{t+j} \frac{\tilde{c}(\alpha)}{\tilde{p}(\alpha)} a(j) = \alpha^t \frac{\tilde{c}(\alpha)}{\tilde{p}(\alpha)} \left(\sum_{j=0}^{T-t} \alpha^j p(j) \right),$$

so that the right hand side of the inequalities in (46) can be simplified. □

If (46) is regarded as a dual linear program the primal must be (after substituting the result of problem 31),

$$\text{Minimize} \quad \frac{\tilde{c}(\alpha)}{\tilde{p}(\alpha)} \sum_{t=1}^{T} \alpha^t \left(\sum_{j=0}^{T-t} \alpha^j p(j) \right) g(t)$$

$$\text{(47)} \qquad \text{subject to} \quad \sum_{j=1}^{t} p(t-j)g(j) \geqq b(t)$$

$$g(t) \geqq 0, \qquad 1 \leqq t \leqq T.$$

<u>Problem 32</u>: Relate (47) to the linear program (4) in Section 4.

Problem 33: Let z^* be the optimal value of the objective function in (47). Although z^* is not differentiable with respect to the right-hand-side of any of the constraints in (47), we act as though it is and write $\frac{\partial z^*}{\partial b(t)}$ for the rate of change of the optimal value of the objective function with a change in the net requirements in period t. This we call the marginal cost (MC) of requirements in period t. Show that $\frac{\partial z^*}{\partial b(t)} = v^*(t)$, the optimal value of the t-th variable in (46).

Problem 34: Continuation of Problem 30.

Write the linear program (47) for the effectiveness model.

Problem 35: Continuation of Problem 16.

Suppose $x(t)$ is the lower bound on accessions in period t. If z^* is the optimal value of (47), show

$$\frac{\partial z^*}{\partial x(t)} = \alpha^t \tilde{c}(\alpha) - \sum_{j=0}^{\infty} v^*(t+j)p(j).$$

Suppose z^* is the optimal value of (47) plus any fixed costs (independent of $\{g(t)\}$). Then calculate $\frac{\partial z^*}{\partial x(t)}$.

Problem 36: Let $s(0) = [s_0(0),\ldots,s_M(0)]$ be the length of service distribution at time 0. The value of $s(0)$ affects the cost legacy and the requirements legacy. Let z^* be the optimal value of (47) plus the present value of the cost legacy. Find $\frac{\partial z^*}{\partial s_j(0)}$ and interpret the result.

Problem 37: To smooth the flow of accessions we can charge a premium λ on all accessions above a certain level h. The infinite horizon problem becomes

$$\text{Minimize} \quad \tilde{c}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t) + \lambda \sum_{t=1}^{\infty} \alpha^t y(t)$$

$$\text{subject to} \quad \sum_{j=1}^{t} p(t-j)g(j) \geq b(t)$$

$$y(t) - g(t) \geq -h$$

$$y(t) \geq 0, \quad g(t) \geq 0 \qquad t = 1,2,\ldots$$

Suppose $b(t) = b$ for $t \geqq T + 1$. Show that an approximately optimal solution of the infinite problem can be obtained by solving the finite linear program:

$$\text{Minimize} \quad \tilde{c}(\alpha) \sum_{t=1}^{T} \alpha^t g(t) + \tilde{c}(\alpha) g + \lambda \sum_{t=1}^{T} \alpha^t y(t) + \lambda y$$

$$\text{Subject to} \quad \sum_{j=1}^{t} p(t-j) g(j) \geqq b(t)$$

$$\sum_{j=1}^{T} \alpha^j \left(\sum_{u=T+1-j}^{\infty} \alpha^u p(u) \right) g(j) + \tilde{p}(\alpha) g \geqq \frac{\alpha^{T+1} g}{1-\alpha}$$

$$-g(t) + y(t) \geqq -h$$

$$-g + y \geqq - \frac{\alpha^{T+1} h}{1-\alpha}$$

$$g(t), g, y(t), y \geqq 0 \qquad t = 1,2,\ldots,T.$$

□

To end this section we consider the problem of maximizing a measure of effectiveness subject to budget constraints. Let $b(t)$ be the net budget constraint at time t that is the budget minus cost legacy. If we maximize a weighted sum of future strength, with weight $\alpha^{t-1}/(1-\alpha)$ applied at time t, and measure strength by stock level $s(t)$, then the problem is equivalent to

$$\text{Maximum} \quad \tilde{p}(\alpha) \sum_{t=1}^{\infty} \alpha^t g(t)$$

$$\text{(48)} \qquad \text{subject to} \quad \sum_{j=1}^{t} c(t-j) p(t-j) g(j) \leqq b(t)$$

$$g(t) \geqq 0,$$

$$\text{for} \quad t \geqq 1.$$

The assumptions for this problem are:

(i) $c(j) > 0$

(49) (ii) $\tilde{p}(\alpha) > 0$

(iii) $\sum_{t=1}^{\infty} \alpha^t b(t) < \infty$

(iv) $b(t) \geqq 0$ for all $t \geqq 1$

(v) $p(u) = 0$ for $u > \mu$.

Problem 38:

Rewrite (48) and (49) for the case of maximizing effective strength with lower bounds on the accessions.

The following simple policies are worthy of closer examination:

$$g^{\times}(t) = \frac{b(t) - \sum_{j=1}^{t} c(t-j)p(t-j)g^{\times}(j)}{c(0)p(0)}, \tag{50}$$

and

$$g^{+}(t) = \max_{0 \leq u \leq M} \left[\frac{b(t+u) - \sum_{j=1}^{t-1} c(t+u-j)p(t+u-j)g^{+}(j)}{c(u)p(u)} \right]. \tag{51}$$

The solution $\{g^{\times}(t)\}$ exactly meets the budget constraints but, there is no guarantee that it will be non-negative. The solution $g(t)$ is non-negative and looks M periods ahead to make $g(t)$ as large as possible without exhausting the remaining budget in years t, $t+1,\ldots,t+M$.

The dual of (48) is

$$\text{Minimize} \quad \sum_{t=1}^{\infty} v(t)b(t)$$

$$\text{subject to} \quad \sum_{j=0}^{\infty} v(t+j)\, c(j)p(j) \geqq \alpha^t \tilde{p}(\alpha),$$

$$v(t) \geqq 0, \qquad t \geq 1.$$

A dual feasible solution is

(53) $$v^{\times}(t) = \alpha^t \tilde{p}(\alpha)/\tilde{c}(\alpha)$$

where $\tilde{c}(\alpha) = \sum_{u=0}^{\infty} \alpha^u c(u)p(u)$. The results for this problem are analogous to problem (33) and (35). They are summed up in the following:

Theorem 4. Under assumption (49).

(i) Both (48) and (52) have optimal solutions $\{g^*(t)\}$ and $\{v^*(t)\}$. Moreover

$$\sum_{t=1}^{\infty} v^*(t)b(t) = \tilde{p}(\alpha) \sum_{t=1}^{\infty} \alpha^t g^*(t).$$

(ii) If

$$\frac{b(t+1)}{b(t)} \geqq \frac{c(j)p(j)}{c(j-1)p(j-1)} \quad \text{for all} \quad 1 \leqq j \leqq t.$$

then $\{g^{\times}(t)\}$ which solves (50) is non-negative and optimal.

(iii) If $c(j)p(j) > \alpha c(j+1)p(j+1)$ for all $j \geqq 0$, then $\{g^{+}(t)\}$ which solves (51) is optimal.

(iv) Approximately optimal solutions can be found by solving the linear program:

$$\text{Maximize} \quad \frac{\tilde{p}(\alpha)}{\tilde{c}(\alpha)} \sum_{t=1}^{T} \alpha^t \left(\sum_{j=1}^{T-t} \alpha^j c(j)p(j) \right) g(t)$$

$$\text{subject to} \quad \sum_{j=1}^{t} c(t-j)p(t-j)g(j) \geqq b(t)$$

$$g(t) \geqq 0 \qquad t = 1,2,\ldots,T.$$

6. Uncertain Requirements.

In most manpower planning situations there is a great deal of uncertainty as to what conditions will prevail in future periods, and different conditions can affect both future manpower requirements and flows. For example, a change in economic conditions affects the demand for automobiles, and a change in this demand causes a change in the manpower requirements of auto manufacturers. If this this requirement decreases it usually affects the natural mobility of the labor force. People usually tend to stay longer in a given job when adverse economic conditions prevail, and this change in mobility is reflected in the flow fractions, the matrix Q in a cross-sectional model, or the matrices $P(u)$ in a longitudinal model. Since future uncertainty is the rule rather than the exception it is highly desirable that we develop models which take it into account. In this section we present a technique for calculating "approximately optimal" long run policies under uncertainty. The technique is similar to that presented in section 2. Since the material in this section is necessarily technical the theory is presented in its simplest form. The methods can be extended and embellished.

With the introduction of uncertainty it behooves us to use a cross-sectional model. Thus, we assume there are N manpower classes in the system, and $S(t)$ is a random N vector which gives the stocks at time t. We introduce the important concept of "conditions" which prevail at some time t. It is assumed that the manpower system is operating under one of K sets of conditions at any time. The conditions at time t are denoted by the random variable $X(t)$ which can take on values 1 through K. Conditions can change from period to period, but clearly it is reasonable to assume that the conditions prevailing at some time t are dependent on the conditions prevailing at time $t-1$. We assume the dependence is Markovian and let

$$(46) \qquad p_{\ell k} = P[X(t+1) = k \mid X(t) = \ell], \qquad \ell, k = 1,2,\ldots,K.$$

Given the condition at time zero the probability of any particular condition prevailing at time t is completely determined by the $K \times K$ matrix P whose (ℓ,k)-th element is $p_{\ell k}$.

Given that conditions k prevail at time t, (i.e. that $X(t) = k$), the fractional flows are given by an $N \times N$ matrix Q_k, the requirements by an N vector r_k, and the input flows in period $t+1$ by $F(t+1,k)$. Notice that the subscript 0 on the input flows used in earlier chapters is dropped here to simplify the notation.

In terms of random variables the stocks at some time $t+1$ are given by

$$(47) \qquad S(t+1) = Q_{X(t)} S(t) + F(t+1).$$

Let $R(t)$ be the random vector of requirements at time t and let

$$(48) \qquad Y(t) = \mathrm{Max}[0, R(t) - S(t)],$$

$$(49) \qquad Z(t) = \mathrm{Max}[0, S(t) - R(t)].$$

Here the maximum is taken element by element in the N vectors. The interpretation is that the i-th element of $Y(t)$ gives the deficit in class i at time t (requirements not met in class i) and that of $Z(t)$ gives the surplus (requirements exceeded in class i). Clearly not both can be positive. Costs are imposed at time t on (a) the stocks, (b) the deficits, (c) the surpluses, and (d) the input flows. Future costs are discounted at rate α. The problem is to find input flows in period 1 in order to minimize the total expected future costs, given the starting stocks $S(0) = s$ and starting conditions $X(0) = k$. This problem is extremely difficult to solve exactly. Since the cross-sectional model is at best an approximation to a real manpower system it seems reasonable to search for an approximation to this optimal solution, one which is easy to calculate. This is done in two stages. The first stage is the formulation of an infinite horizon linear program whose optimal solution gives a lower bound for the problem stated above. The second stage uses techniques described in section 2 to reduce this to a small finite linear program which is easily solved and from which can be extracted a reasonable and good operating policy.

Equation (46) gives the 1-step "forward" probabilities for the changes in conditions. In stage 1 we need the 1-step "backward" probabilities, $P[X(t)=k \mid X(t+1)=\ell]$. Let $\pi_k(\ ) = P[X(t)=k]$, which is easily found from P and the starting condition. An application of Bayes' Law gives

$$P[X(t)=k \mid X(t+1)=\ell] = \pi_k(t)p_{k\ell}/\pi_\ell(t+1), \tag{50}$$

where if both numerator and denominator are zero the quotient is taken to be zero.

Now define

$$S(t,k) = E[S(t) \mid X(t)=k]$$

and

$$F(t+1,k) = E[F(t+1) \mid X(t)=k].$$

Using these with (50), (47) and conditional expectations,

$$S(t+1,k) = \sum_{\ell=1}^{K} [\pi_\ell(t)p_{\ell k}/\pi_k(t+1)]\ [Q_\ell S(t,\ell)+F(t+1,\ell)].$$

To simplify this expression let

$$s(t,k) = \pi_k(t)S(t,k)$$

and

$$f(t+1,k) = \pi_k(t)F(t+1,k).$$

Then we have

$$s(t+1,k) = \sum_{=1}^{K} p_{\ell k}Q_\ell s(t,\ell) + \sum_{+L}^{K} p_{\ell k}f(t+1,\ell). \tag{51}$$

Any sequence of stocks and flows must satisfy this equation at each time point.

Now let

$$Y(t,k) = E[Y(t) \mid X(t) = k],$$

$$Z(t,k) = E[Z(t) \mid X(t) = k],$$

$$y(t,k) = \pi_k(t)Y(t,k),$$

and

$$z(t,k) = \pi_k(t)Z(t,k).$$

Combining (48) and (49) we see that

$$Y(t) - Z(t) = K(t) - S(t),$$

and by using conditional expectation and the above definitions,

$$s(t,\ell) + y(t,\ell) - z(t,\ell) = \pi_\ell(t)r_\ell. \tag{52}$$

Any sequence of stocks, surpluses, deficits and requirements must satisfy this equation for each t.

A policy is a rule which specifies for all $t \geq 0$ a value of the input flows, surpluses and deficits, given the stocks and conditions. Thus, a feasible policy must satisfy both (51) and (52). In order to discuss optimal policies we need to introduce costs.

Let a_ℓ, b_ℓ, c_ℓ and d_ℓ be N vectors of single period unit costs for stocks, input flows, deficits and surpluses respectively, and let α be the discount factor. Then the present value of the total expected costs over the infinite horizon is

$$\sum_{t=1}^{\infty} \alpha^t \sum_{\ell=1}^{K} [a_\ell s(t,\ell) + b_\ell f(t,\ell) + c_\ell z(t,\ell) + d_\ell y(t,\ell)], \tag{53}$$

Our stage 1 problem is now complete: Find N vectors $f(t,\ell)$, $z(t,\ell)$ and $y(t,\ell)$, $t \geq 1$, $\ell = 1,2,\ldots,K$, which are non-negative, satisfy (51) and (52), and minimize (53).

Problem 31: Show that the solution to the infinite horizon linear program above gives a lower bound on the minimum expected cost of the original problem. To do this show that a feasible policy in the original problem must satisfy constraints in addition to those given in (51) and (52) and non-negativity. □

This stage 1 problem is itself very difficult to solve, and we proceed to stage 2 to approximate it with a finite linear program. Following the ideas presented in section 2, define

$$s_\ell = \sum_{t=1}^{\infty} \alpha^t s(t,\ell), \qquad f_\ell = \sum_{t=1}^{\infty} \alpha^t f(t,\ell),$$

$$y_\ell = \sum_{t=1}^{\infty} \alpha^t y(t,\ell), \qquad z_\ell = \sum_{t=1}^{\infty} \alpha^t z(t,\ell).$$

Using these definitions the objective function in (53) becomes

(54) $$\sum_{\ell=1}^{K} [a_\ell s_\ell + b_\ell f_\ell + c_\ell z_\ell + d_\ell y_\ell].$$

After multiplication by α^{t+1} and summing over t, (51) becomes

(55) $$s_k = \sum_{\ell=1}^{K} \alpha p_{\ell k} Q_k s_\ell + \sum_{\ell=1}^{K} p_{\ell k} f_\ell + \alpha \sum_{\ell=1}^{K} p_{\ell k} s(0,\ell).$$

Given the starting conditions are $s(0) = s$ and $X(0) = k$, then

$$\begin{aligned} s(0,\ell) &= 0 \quad \text{if} \quad \ell \neq k, \\ &= s \quad \text{if} \quad \ell = k. \end{aligned}$$

After multiplication by α^t and summing over t, (52) becomes

(56) $$s_\ell + y_\ell - z_\ell = \tilde{\pi}_\ell(\alpha) r_\ell,$$

where

(57) $$\tilde{\pi}_\ell(\alpha) = \sum_{t=1}^{\infty} \alpha^t \pi_\ell(t).$$

Our stage 2 problem is: Find N vectors f_ℓ, y_ℓ and z_ℓ, $\ell = 1,2,\ldots,K$ which are non-negative, satisfy (55) and (56), and minimize (54). The variables s_ℓ are determined by (55).

In order to simplify and analyze the stage 2 program further we introduce the following notation:

$$Q = \begin{bmatrix} p_{11}Q_1 & p_{21}Q_2 & \cdots & p_{K1}Q_K \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ p_{1K}Q_1 & p_{2N}Q_2 & \cdots & p_{KK}Q_K \end{bmatrix},$$

an NK by NK matrix, and

$$\tilde{r} = [\tilde{\pi}_1(\alpha)r_1,\ldots,\tilde{\pi}_K(\alpha)r_K],$$

$$\tilde{s}(0) = [s(0,1),\ldots,s(0,K)],$$

$$\tilde{s} = [s_1,\ldots,s_K],$$

$$f = [f_1,\ldots,f_K],$$

$$y = [y_1,\ldots,y_K],$$

$$z = [z_1,\ldots,z_K],$$

all NK column vectors,

$$a = (a_1,\ldots,a_K),$$

$$b = (b_1,\ldots,b_K),$$

$$c = (c_1,\ldots,c_K),$$

$$d = (d_1,\ldots,d_K),$$

all NK row vectors. Also let

$$H = \begin{bmatrix} p_{11}I & \cdots & p_{K1}I \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ p_{1K}I & \cdots & p_{KK}I \end{bmatrix},$$

an NK by NK matrix, where I represents an N by N identity matrix.

The stage 2 linear program can now be written as

$$\begin{aligned} &\text{Minimize} \quad a\tilde{s} + bf + cz + dy \\ (58) \qquad &\text{subject to} \quad \tilde{s} = \alpha Q\tilde{s} + HF + \alpha H\tilde{s}(0), \\ &\tilde{s} + y - z = \tilde{r}, \\ &\tilde{s} \geq 0 \\ &y \geq 0 \\ &z \geq 0 \\ &f \geq 0. \end{aligned}$$

The first set of constraints in (58) can be solved for $\tilde{s}$, giving

$$\tilde{s} = \tilde{Q}(\alpha)H[f + \alpha\tilde{s}(0)],$$

where $\tilde{Q}(\alpha) = (I - \alpha Q)^{-1}$.

Since $\tilde{Q}(\alpha)$ is non-negative, $\tilde{s}$ is non-negative and (58) can be simplified to give:

$$\begin{aligned} &\text{Minimize} \quad [a\tilde{Q}(\alpha)H + b]f + cz + dy \\ (59) \qquad &\text{subject to} \quad \tilde{Q}(\alpha)HF + y - z = \tilde{r} - \alpha\tilde{Q}(\alpha)H\tilde{s}(0), \\ &y \geq 0, \quad z \geq 0, \quad g \geq 0. \end{aligned}$$

The linear program (59) has 3NK variables and NK constraints. From the basic flow data $\{Q_k\}$, the requirement data $\{r_k\}$, the "condition" transition probabilities P, the starting conditions X(0) and $\tilde{s}(0)$, and the discount factor α, these constraints can be written out. The coefficients of the objective function are determined by this data and the data on costs, namely a, b, c, and d.

Suppose that (59) is solved and let (f^*,y^*,z^*) be the optimal value of the variables. The problem remains of how to obtain a "policy" from this solution; given the conditions at time t, what input flows should be made in period $t+1$? We now show how to obtain a stationary policy from f^*. This is a policy which gives the input flows in the next period for each current condition. These flows depend only on the condition and not on the actual period.

Recall that f is the column vector $[f_1,f_2,\ldots,f_K]$, where $f_k = \sum_{t=1}^{\infty} \alpha^t f(t,k)$, and $f(t,k) = \pi_k(t-1)F(t,k)$. Our stationary policy assumption implies $F(t,k) = F_k$, independent of t. F_k is an N vector which gives the input flows in the next period if the current conditions are k. Using these definitions we obtain

$$f_k = \sum_{t=1}^{\infty} \alpha^t \pi_k(t-1)\; F_k,$$

and from (57),

$$f_k = \alpha(\pi_k(0) + \tilde{\pi}_k(\alpha))F_k.$$

Thus our stationary policy becomes

$$F_k^* = [\pi_k(0) + \tilde{\pi}_k(\alpha)]^{-1} \cdot f_k^*, \qquad k = 1,2,\ldots,K. \tag{50}$$

Problem 32: From Markov chain theory the vector $\pi(t)$ with k-th element $\pi_k(t)$ is given by $\pi(t) = \pi(0)P^t$. Show that

$$\tilde{\pi}(\alpha) = \pi(0)[I-\alpha P]^{-1} - \pi(0).$$

□

We end this section with a simple numerical example to illustrate the use of the linear program (59). Suppose there are two manpower classes and two possible conditions $(N=K=2)$, with fractional flow matrices

$$Q_1 = \begin{bmatrix} .8 & 0 \\ .1 & .8 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} .6 & 0 \\ .2 & .6 \end{bmatrix}.$$

The condition transition matrix is

$$P = \begin{bmatrix} .5 & .5 \\ .4 & .6 \end{bmatrix}.$$

The starting conditions are,

$$s(0) = [100,100], \quad \text{and} \quad X(0) = 1.$$

The requirements are

$$r_1 = [105,90] \quad \text{and} \quad r_2 = [100,80].$$

The costs are

$$a_1 = b_1 = (1,1), \qquad a_2 = b_2 = (2,2),$$

$$c_1 = (20,30), \qquad c_2 = (15,23),$$

$$d_1 = (52,10), \qquad d_2 = (2,34).$$

Finally, the discount factor $\alpha = 0.9$.

Using this data the reader can check that

$$Q = \begin{bmatrix} 0.40 & & 0.24 & \\ 0.05 & 0.40 & 0.08 & 0.24 \\ 0.40 & & 0.36 & \\ 0.05 & 0.04 & 0.12 & 0.36 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.5 & & 0.4 & \\ & 0.5 & & 0.4 \\ 0.5 & & 0.6 & \\ & 0.5 & & 0.6 \end{bmatrix},$$

$$\tilde{Q}(\alpha)H = \begin{bmatrix} 1.257 & & 1.127 & \\ 0.428 & 1.257 & 0.429 & 1.127 \\ 1.409 & & 1.488 & \\ 0.537 & 1.409 & 0.542 & 1.488 \end{bmatrix},$$

$$a\tilde{Q}(\alpha)H + b = (6.57,\ \ 5.07,\ \ 7.61,\ \ 6.10),$$

and

$$\tilde{r} - \alpha\tilde{Q}(\alpha)Hs(0) = [312.7,\ 213.3,\ 367.7,\ 220.5].$$

These numbers are used in a linear program (59) with four constraints and 12 variables. From the optimal solution of this program we obtain

$$f^* = [180,\ 0,\ 76,\ 72],$$

and by using (60) we obtain the stationary policy

(i) If in condition 1 at t, input flows in period $t+1$ are [40, 0],

(ii) If in condition 2 at t, input flows in period $t+1$ are [17, 16].

7. Notes and Comments.

This chapter presents some applications of optimization procedures to manpower flow models. As stated in its introduction, there are very many ways to use optimization procedures in manpower modelling, so one cannot present a single proper or correct procedure. The value of any approach will always depend on the context of the manpower system and the objectives of the planners and policy makers.

Several other books contain articles on optimization in manpower systems. These include Bartholomew [1973], Charnes, Cooper, and Neihaus [1972], Smith [1971], and Bartholomew and Morris [1971].

Section 2 is based on Grinold [1974a], and Grinold and Stanford [1974]. Section 3 presents a novel approach to system design. Section 4 is based on Grinold and Stanford [1974] and Grinold and Hopkins [1973]. Section 5 is drawn from Grinold, Marshall, and Oliver [1973]. Finally section 6 is derived from Grinold [1973], [1974b] and [1974c].

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